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Matrix model and time-like linear dilaton matter

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Abstract: We consider a matrix model description of the 2d string theory whose matter part is given by a time-like linear dilaton CFT. This is equivalent to the $c = 1$ matrix model with a deformed, but very simple Fermi surface. Indeed, after a Lorentz transformation, the corresponding 2d spacetime is a conventional linear dilaton background with a time-dependent tachyon field. We show that the tree level scattering amplitudes in the matrix model perfectly agree with those computed in the world-sheet theory. The classical trajectories of fermions correspond to the decaying D-branes in the time-like linear dilaton CFT. We also discuss the ground ring structure. Furthermore, we study the properties of the time-like Liouville theory by applying this matrix model description. We find that its ground ring structure is very similar to that of the minimal string.

Keywords: Conformal Field Models in String Theory, Tachyon Condensation, 2D Gravity, Matrix Models.

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1. Introduction

It is well-known that the two dimensional string theory with a static linear dilaton and Liouville potential can be described non-perturbatively by the dual matrix model [1]–[3], called $c = 1$ matrix model.[1] At the world-sheet level, this model is equivalent to a free boson theory (with the central charge $c = 1$ matter) plus the Liouville theory ($c = 25$), defined by the world-sheet action[2] and the string coupling constant

$$S = \int d\sigma^2[-\partial X^0\bar{\partial}X^0 + \partial\phi\bar{\partial}\phi + \mu e^{2\phi}]\,, \qquad g_s = e^{2\phi}\,. \tag{1.1}$$

There is only one propagating scalar field η, which is related to the tachyon field T in bosonic string via $T \sim g_s \cdot \eta$. It behaves like a massless scalar field in the 2d linear dilaton background. The dual $c = 1$ matrix model is defined by a quantum mechanics of a $N \times N$ hermitean matrix Φ with a inverse harmonic potential (after the double scaling

[1] For reviews see e.g. [4]–[8].
[2] In this paper we set $\alpha' = 1$.

limit $N \to \infty$)

$$S_{\rm mat} = \int dt\, {\rm Tr}\left[(D_t\Phi)^2 + \Phi^2\right]. \tag{1.2}$$

Here, $D_t = \partial_t - i[A_t,]$ denotes the covariant derivative with respects to the $U(N)$ gauge symmetry, projecting out non-singlet sectors. The eigenvalues x of Φ behave like N free fermions and they form a Fermi sea. The static vacuum (1.1) of string theory corresponds to the static Fermi surface

$$p^2 - x^2 = -2\mu\,, \tag{1.3}$$

in the two dimensional semiclassical phase space $(x, p) \equiv (x, \dot{x})$. We can also employ the type 0 model [9, 10] or type II model [11, 12] to make the non-perturbative issues clearer.

As a next step, it will also be natural and interesting to ask what will happen if we consider a spacetime with a different property in the time direction. One of the simplest examples will be the time-like linear dilaton theory and this is a basic example of time-dependent backgrounds in string theory.[3] In our context, we can consider a string model defined by the time-like linear dilaton theory (with the central charge $c = 1 - 6q^2$) plus the space-like Liouville theory ($c = 1 + 6Q^2$) on the world-sheet. This may be called a non-minimal $c < 1$ non-critical string. Its world-sheet action is simply given by

$$S = \int d\sigma^2 \left[-\partial X^0 \bar{\partial} X^0 + \partial\phi\bar{\partial}\phi + \mu e^{2b\phi}\right], \tag{1.4}$$

with the background charge terms which correspond to the coupling constant

$$g_s = e^{qX_0 + Q\phi}\,. \tag{1.5}$$

The values of the background charges are

$$Q = b + \frac{1}{b}\,, \qquad q = -b + \frac{1}{b}\,, \tag{1.6}$$

in terms of the parameter b, which satisfies the condition[4]

$$0 < b < 1\,. \tag{1.7}$$

In this rather simple example we can solve the theory exactly by applying known results of the Liouville theory [20]–[22]. It is obvious that the system will get strongly coupled in the late time. However, if we consider the physical process of scattering of closed strings from the Liouville wall, the process itself does not occur in the strongly coupled region because of the inequality $q < Q$.

After the Lorentz transformation,

$$\tilde{X^0} = \frac{Q}{2}X^0 + \frac{q}{2}\phi\,, \qquad \tilde{\phi} = \frac{q}{2}X^0 + \frac{Q}{2}\phi, \tag{1.8}$$

[3]Refer to e.g. [13]–[18] for recent discussions.

[4]Here, the condition $b < 1$ comes from the Seiberg bound [19] and also we assumed that b is positive using the sign flip $\phi \to -\phi$.

we can equivalently obtain the usual static linear dilaton vacuum perturbed by a time-dependent Liouville potential defined by

$$S=\int d\sigma^2\left[-\partial\tilde{X^0}\bar{\partial}\tilde{X^0}+\partial\tilde{\phi}\bar{\partial}\tilde{\phi}+\mu\exp\Big((b^2-1)\tilde{X^0}+(1+b^2)\tilde{\phi}\Big)\right],\qquad g_s=e^{2\tilde{\phi}}. \quad (1.9)$$

In general, time-dependent backgrounds in 2d string theory correspond to deformed and time-dependent Fermi surfaces in the $c=1$ matrix model and this issue has been discussed in the papers [23]–[30]. As recently pointed out in [28], they lead to non-perturbatively tractable examples of the interesting time-dependent model of closed string tachyon condensation. In this paper we would like to closely understand the duality between the time-dependent backgrounds in 2d string theory and the matrix model with a deformed Fermi surfaces via the special example (1.9), where we can solve the theory in both sides.

It is also intriguing to consider the case where b is imaginary (or $b^2<0$). This corresponds to the time-like Liouville theory [14, 15, 31] after the double wick rotation $(X^0,\phi)\rightarrow(-i\phi,-iX^0)$ in (1.4). Since this conformal field theory is far from well-understood, the matrix model formulation should be definitely useful. As we will see later, indeed we find rather different properties compared with those in the usual space-like Liouville theory.

This paper is organized as follows. In section 2 we first give a direct matrix model dual of the 2d string theory with the time-like linear dilaton matter; and then we show that the model is equivalent to the ordinary $c=1$ matrix model via a field redefinition as expected from the Lorentz invariance. We also compute the closed string emission from the decaying D-branes and identify the leg factor from the results. In section 3 we give an equivalent description as a time-dependent background in $c=1$ matrix model. We also compute the scattering S-matrices in this background and find agreements with those in the world-sheet theory. In section 4 we discuss the time-like Liouville theory by applying the matrix model dual. We correctly reproduce the expected spacetime geometry using the collective field description. In section 5 we consider the ground ring structure of our background and discuss the relation to non-compact Calabi-Yau manifolds. In section 6 we summarize the results and discuss future problems.

2. Matrix model and 2D string with time-like linear dilaton matter

First, let us try to derive directly the matrix model dual of the 2d string with the time-like linear dilaton matter defined by (1.4) and (1.5). To construct a matrix model for a new background it is helpful to remember the recent interpretation of the $c=1$ matrix model as a theory of unstable D0-branes (so called ZZ-brane [32]) [33, 34]. The matrix Φ can be regarded as a open-string tachyon field on them and the matrix model itself corresponds to an effective action of such D-branes. Then we can argue that a matrix model dual of time-like linear dilaton background (1.4) is defined by

$$S_{mat'}=\int dt\ e^{-qt}\,\mathrm{Tr}\left[(D_t\Phi)^2+\Phi^2\right]. \quad (2.1)$$

We have put the time-dependent factor e^{-qt} because the D-brane action is proportional to $g_s^{-1} \propto e^{-qt}$ under the identification $X^0 = t$. We chose the tachyonic mass term in (2.1) such that it agrees with the mass of the D0-brane [32] calculated in the boundary Liouville theory.

By using the gauge symmetry we can again diagonalize the matrix into the eigenvalues λ_i. Then the action becomes

$$S_{\mathrm{mat}'} = \int dt\ e^{-qt} \sum_i \left[\dot{\lambda}_i(t)^2 + \lambda_i(t)^2\right] . \tag{2.2}$$

The classical trajectories in this system (2.2) are given by

$$\lambda(t) = C_1 e^{-bt} + C_2 e^{\frac{1}{b}t} , \tag{2.3}$$

where C_1 and C_2 are arbitrary constants. They correspond to the time-dependent open string tachyon field (so called Rolling tachyon [35]) on unstable D0-branes.

Actually, after the redefinition of the variable

$$\lambda_i(t) = e^{\frac{q}{2}t} x_i(t) , \tag{2.4}$$

the action can be written as (up to total derivative terms)

$$S_{\mathrm{mat}'} = \int dt\ \sum_i \left[\dot{x}_i(t)^2 + \left(1 + \frac{q^2}{4}\right) x_i(t)^2\right] . \tag{2.5}$$

Now we have the conventional $c = 1$ matrix model with a shifted tachyon mass. This is expected since we know that the 2d string with the time-like linear dilaton matter is equivalent to the conventional $c = 1$ string via the Lorentz transformation (1.8). Indeed if we perform the Lorentz transformation $\frac{\partial}{\partial t} = \frac{Q}{2}\frac{\partial}{\partial \tilde{t}} + \frac{q}{2}\frac{\partial}{\partial \tilde{\phi}}$ into the usual two dimensional string theory with the linear dilaton (1.9) , then we can derive the ordinary action of $c = 1$ matrix model with the correct tachyon mass

$$S_{\mathrm{mat}} = \int d\tilde{t}\ \sum_i \left[\dot{x}_i(\tilde{t})^2 + x_i(\tilde{t})^2\right] . \tag{2.6}$$

2.1 Closed string emission and leg factor

From the world-sheet viewpoint, a particular class of open string tachyon condensation on the unstable D0-brane can be represented by the boundary time-like Liouville theory [36]–[40] (so called the half S-brane) defined by the action

$$S = \int d\sigma^2 (-\partial X^0 \bar{\partial} X^0) + \mu_B \int_{\partial\Sigma} d\sigma e^{-bX^0} , \tag{2.7}$$

corresponding to the first term in (2.3). The second term in (2.3) is explained as the dual boundary cosmological constant. This theory can be regarded as a time-like continuation [38, 41] of the boundary conformal field theory on a FZZT-brane [42, 43]. By using this observation, we can compute the closed string one-point function on the decaying

D0-brane as follows[5]

$$\left\langle e^{(q-iE)X^0+(Q+iP)\phi}\right\rangle_{E=P} = e^{-i\frac{E}{b}\log\mu_B}\cdot(\mu\gamma(b^2))^{-i\frac{P}{2b}}\cdot\frac{\Gamma(iP/b)}{\Gamma(-iP/b)}$$
$$\left\langle e^{(q-iE)X^0+(Q+iP)\phi}\right\rangle_{E=-P} = e^{-i\frac{E}{b}\log\mu_B}\cdot(\mu\gamma(b^2))^{-i\frac{P}{2b}}\cdot\frac{\Gamma(iPb)}{\Gamma(-iPb)}\,, \qquad (2.8)$$

where we have defined $\gamma(b^2) = \frac{\Gamma(b^2)}{\Gamma(1-b^2)}$; the on-shell conditions are given by $E = P$ and $E = -P$ in the above two cases, respectively.

The physical meaning of this one-point function is the closed string emission from the decaying D-branes [44]. In 2d string theory, the closed string field is equivalent to the fluctuation of the Fermi surface via the bosonization up to the momentum dependent phase factor called the leg-factor. On the other hand, each fermion itself can be regarded as a decaying D0-brane [33, 34]. As pointed out in [34], we can directly confirm these identifications from the fact that the closed string emission amplitude is given by a phase factor which coincides with the leg-factor (except the energy dependent term due to the time-delay). Interestingly, we can also find a similar story in our generalized backgrounds (1.4) (1.5). Indeed the closed string emission (2.8) is given by a pure phase factor. Furthermore, we can check that it is the same as the leg-factor. To see this, consider the two point function (or reflection coefficient) [20]–[22]

$$S(P) \equiv \left\langle e^{(q-iP)X^0+(Q+iP)\phi}\; e^{(q+iP)X^0+(Q+iP)\phi}\right\rangle = -(\mu\gamma(b^2))^{-iP/b}\frac{\Gamma(iP/b)\Gamma(ibP)}{\Gamma(-iP/b)\Gamma(-ibP)}\,. \qquad (2.9)$$

This is exactly the multiplication of the two terms in (2.8).

These discussions on the closed string emission and leg-factor can be made clearer by performing the Lorentz transformation (1.8) of these quantities into the system (1.9). The transformed energy and momentum are given[6] by

$$\tilde{E} = \frac{Q}{2}E + \frac{q}{2}P\,, \qquad \tilde{P} = \frac{q}{2}E + \frac{Q}{2}P\,. \qquad (2.10)$$

Then we find the closed emission from the half s-brane

$$\left\langle e^{-i\tilde{E}\tilde{X}^0+(2+i\tilde{P})\tilde{\phi}}\right\rangle_{\tilde{E}=\tilde{P}} = e^{-i\tilde{E}\log\mu_B}(\mu\gamma(b^2))^{-i\frac{\tilde{P}}{2}}\frac{\Gamma(i\tilde{P})}{\Gamma(-i\tilde{P})} \equiv e^{-i\tilde{E}\log\mu_B}\cdot e^{i\varphi_+(\tilde{P})}\,,$$
$$\left\langle e^{-i\tilde{E}\tilde{X}^0+(2+i\tilde{P})\tilde{\phi}}\right\rangle_{\tilde{E}=-\tilde{P}} = e^{-i\frac{\tilde{E}}{b^2}\log\mu_B}(\mu\gamma(b^2))^{-i\frac{\tilde{P}}{2b^2}}\frac{\Gamma(i\tilde{P})}{\Gamma(-i\tilde{P})} \equiv e^{-i\frac{\tilde{E}}{b^2}\log\mu_B}\cdot e^{i\varphi_-(\tilde{P})}\,. \qquad (2.11)$$

The phase factor $e^{i\varphi_\pm(\tilde{P})}$ should be regarded as the leg factor in our time-dependent background of 2d string. Notice that φ_+ is the same as the usual leg-factor in $c = 1$ matrix model as is expected since we can write the Liouville term in (1.9) as $\mu\exp(2\tilde{\phi})$ when

[5] We assume $\alpha' = 1$ and define the momentum P and energy E such that $p^\mu = (E, P)$ and $p_\mu = (E, -P)$. The vertex operator is given by $e^{(q-iE)X^0+(Q+iP)\phi}$. When $P > 0$ the particle is moving toward strongly coupled region $\phi \to \infty$. Also, in the computation of correlators, we are using a slightly different normalization of μ (by the factor π) compared with the paper [42].

[6] In particular, for a massless particle with $E = P$ we have the relation $\tilde{E} = \tilde{P} = E/b = P/b$, while in the opposite case $E = -P$ we get $\tilde{E} = -\tilde{P} = bE = -bP$.

$\tilde{X^0} = -\tilde{\phi}$. When we consider a incoming wave with the energy $\tilde{E}(= \tilde{P})$ and its reflection, the energy of the outgoing wave is shifted into $\tilde{E}' = b^2\tilde{E}(= -\tilde{P}')$ due to the Doppler shift since the the Liouville wall (1.9) is moving. Then the two point function is given by

$$S(\tilde{P}) \equiv \left\langle e^{-i\tilde{P}\tilde{X}^0+(2+i\tilde{P})\tilde{\phi}}\ e^{ib^2\tilde{P}\tilde{X}^0+(2+ib^2\tilde{P})\tilde{\phi}} \right\rangle = -(\mu\gamma(b^2))^{-i\tilde{P}} \frac{\Gamma(i\tilde{P})\Gamma(ib^2\tilde{P})}{\Gamma(-i\tilde{P})\Gamma(-ib^2\tilde{P})} . \qquad (2.12)$$

This reflection amplitude can be nicely rewritten in terms of the leg factors

$$S(\tilde{P}) = -e^{i\varphi_+(\tilde{P})} \cdot e^{i\varphi_-(b^2\tilde{P})} , \qquad (2.13)$$

as expected. In this way we have confirmed the identification of matrix model fermions with decaying D-branes[7] in our time-dependent backgrounds (1.9). These results of the leg-factor will also be useful later when we compare the scattering amplitudes in the matrix model with the world-sheet computation for arbitrary values of b.

Even though we have examined the special case $C_1 \neq 0$ and $C_2 = 0$ in (2.3) (i.e. half S-brane), it is natural to expect the similar computations can be done for more general C_1 and C_2 (so called full S-brane) as has been done for $b = 1$ case [34] by using the rolling tachyon boundary state [35]. Thus our matrix model here predicts the existence of boundary states for general profiles of (2.3) in the time-like linear dilaton theory and its construction will be an intriguing future problem. Since the trajectory corresponding to the D-brane should be above the Fermi level in the matrix model, we can find a bound $|C_1|^{b^2}|C_2| \leq \mu$, where μ is the Fermi level for our background and will be defined in the next section.

3. Equivalent time-dependent background in $c = 1$ matrix model

As we have seen in the previous section, the matrix model dual of the 2d string background (1.4) can be given by a time-dependent background of $c = 1$ matrix model. The 2d Lorentz transformation is not clear in the holographic dual matrix model since the Liouville direction is hidden inside the infinitely many eigenvalues. Thus it is an non-trivial and intriguing problem to realize the time-dependent background (1.9) in the $c = 1$ matrix model.[8]

We argue that the string theory background (1.9) can be identified with the time dependent Fermi surface in $c = 1$ matrix model (we assume $\mu > 0$)

$$(-p-x)^{b^2}(p-x) = 2^{1+b^2}\mu\ e^{(b^2-1)\tilde{t}} , \qquad (3.1)$$

[7] If we consider the static D0-brane (i.e. ZZ-brane [32] in (1.4)), then naively we will obtain a moving D0-brane at the velocity $q/Q < 1$ in (1.9) after the Lorentz transformation. This D0-brane may not be static since there is the time-dependent Liouville potential. In this matrix model, obviously this configuration corresponds to a single fermion on the top of the inverse harmonic potential.

[8] In most of the literature (e.g. [27]–[30]), a non-zero static cosmological constant as in (1.1) is assumed to compute physical quantities. In particular, it is possible to solve the matrix model for an euclidean compactified time by applying the Toda Lattice integrable structure [45]–[49] for a rather general backgrounds with time-dependent tachyon perturbations as shown in [27] . In the discussions of the present paper, however, we do not put the static cosmological constant term $\sim e^{2\phi}$ (for $b \neq 1$) because it will change the asymptotic behavior and lead to a different theory. Interestingly, this suggests that our backgrounds may be related to a new integrable structure of $c = 1$ matrix model.

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where $\tilde{t}(=\tilde{X^0})$ is the time[9] in the matrix model (see (2.6)). Its qualitative behavior can be summarized as follows. Because of the condition $b^2<1$, in the far past $t\to-\infty$, the Fermi surface is pushed into the infinity and there is no Fermi sea. After that, the Fermi sea gradually begins to appear from the weakly coupled region $|x|>>1$, and it finally spreads out completely. This is intuitively consistent with the property of the time-dependent tachyon field in (1.9). We start with the infinite tachyon condensation, which means that spacetime disappears. Then the tachyon field becomes smaller and the spacetime appears. Eventually, the tachyon field becomes zero and we have the ordinary (strongly coupled) 2d spacetime with the linear dilaton.

In order to see that (3.1) is consistent with the time evolution, we can rewrite it simply as follows

$$W_{1,0}^{b^2}\ W_{0,1}=2^{1+b^2}\mu\,, \tag{3.2}$$

by using the conserved quantities (or the classical w_∞ generators [50, 6]) for each fermion

$$W_{1,0}=-(p+x)e^{-\tilde{t}}\,,\qquad W_{0,1}=(p-x)e^{\tilde{t}}\,. \tag{3.3}$$

Also since we are discussing the 2d bosonic string, only one Fermi surface is relevant and we can only consider the Fermi surface which satisfies the constraints $p+x<0$ and $p-x>0$. The matrix model defined by the two Fermi surfaces, replacing $-p-x$ with $|p+x|$ in (3.1) describes a background with a time-dependent NSNS scalar field in type 0B string theory. Though we mainly restrict to the bosonic string case below, we can obtain the almost same result in the type 0 case as we will briefly comment later.

The profile of the semiclassical Fermi surface includes all information on the dual string theory at tree level, and can be uniquely determined from the action (1.9) as we will see later.[10] To go beyond the tree level we need to define a time-dependent quantum state in the matrix quantum mechanics (1.2) and this problem is beyond the scope of this paper.

In general, when a Fermi surface is given, the expectation value of tachyon field in the asymptotic region $\phi\to-\infty$ can be determined by its deviation from the singular Fermi surface $p^2-x^2=0$. We can write this in the following way,

$$p_\pm\simeq\mp x\pm\frac{\epsilon_\pm}{x}\qquad(x\to-\infty)\,, \tag{3.4}$$

where p_+ (or p_-) is the value of the momentum at the upper (or lower) branch of the Fermi surface (3.1). After identifying the spacial coordinate as $x=e^{-\tilde{\phi}}$, the deviations $\epsilon_\pm$ are related to the left and right-moving part of the massless scalar field η in the 2d spacetime [23] via

$$\begin{aligned}(\partial_{\tilde{t}}-\partial_{\tilde{\phi}})\ \eta(\tilde{t},\tilde{\phi})\ &=\ \pi^{-1/2}\epsilon_+(\tilde{t}-\tilde{\phi}),(\partial_t+\partial_\phi)\\ \eta(\tilde{t},\tilde{\phi})\ &=\ -\pi^{-1/2}\epsilon_-(\tilde{t}+\tilde{\phi})\,.\end{aligned} \tag{3.5}$$

[9]Notice the relation $\tilde{t}\sim\frac{Qt}{2}$ due to the Lorentz transformation.

[10]To be exact, we should say that the cosmological constant μ in (3.1) corresponds to $\mu\gamma(b^2)$ in (1.4) (see appendix A).

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This is explained by the bosonization of Dirac fermions and the massless scalar field η is the collective field of the Fermi sea [51]–[53] and [6, 23]. The scalar field η is related to the tachyon field T in 2d bosonic string as follows

$$T(\tilde{t},\tilde{\phi}) = e^{2\tilde{\phi}} \cdot \eta(\tilde{t},\tilde{\phi})\,, \tag{3.6}$$

up to the leg factor.

Let us apply this method to (3.1) in order to examine the tachyon field in this background. If we expand the Fermi surface near the two asymptotic regions as in (3.5), we get

$$p - x \sim 2\mu e^{(b^2-1)\tilde{t}}|x|^{-b^2}\,, \qquad p + x \sim -2\mu^{1/b^2} e^{(1-1/b^2)\tilde{t}}|x|^{-1/b^2}\,. \tag{3.7}$$

Following the rule (3.5), we can extract the expectation value of the tachyon field from (3.7)

$$T_- = \mu \exp\left((b^2-1)\tilde{X}^0 + (1+b^2)\tilde{\phi}\right), \qquad T_+ = \mu^{1/b^2} \exp\left((1-1/b^2)\tilde{X}^0 + \left(1+\frac{1}{b^2}\right)\tilde{\phi}\right). \tag{3.8}$$

The two tachyon fields T_- and T_+ represent the two contributions from each term in (3.7). The first one T_- exactly coincides with the Liouville potential in (1.9). The second one also agrees with the Lorentz transformation of the dual Liouville potential $\tilde{\mu} e^{\frac{2}{b}\phi}$ ($\tilde{\mu} = \mu^{\frac{1}{b^2}}$). As is known in the Liouville conformal field theory, the dual potential automatically appears whenever we put the original one [20]–[22] and [8]. Thus our matrix model description precisely reproduces this fact. In terms of the dual picture we can also rewrite (3.1) as follows

$$(-p-x)(p-x)^{1/b^2} = \tilde{\mu} e^{(1-1/b^2)\tilde{t}}\,. \tag{3.9}$$

Notice that the form of the Fermi surface (3.1) is determined uniquely by the identification of asymptotic fields and the time-evolution.

From the viewpoint of the matrix model (2.1), which is directly dual to the background (1.4) before the Lorentz transformation, the Fermi surface is given by

$$\left(-\dot{\lambda} - b\lambda\right)^{b^2}\left(\dot{\lambda} - \frac{1}{b}\lambda\right) = 2^{1+b^2}\mu\,. \tag{3.10}$$

This looks like a static background and is consistent with the static Liouville potential in (1.4). To find the result (3.10) , notice that the conserved quantities are now given by

$$W_{1,0} = -(\dot{\lambda} + b\lambda)e^{-t/b}\,, \qquad W_{0,1} = \left(\dot{\lambda} - \frac{\lambda}{b}\right)e^{bt}\,. \tag{3.11}$$

In this way the Lorentz transformation is equivalent to the canonical transformation between the matrix model (2.1) and the usual $c = 1$ matrix model.

3.1 Scattering amplitudes

To find a further evidence that the Fermi surface (3.1) is dual to the background (1.4) or equally (1.9), it is useful to compare the scattering S-matrices. To compute the scattering amplitudes in the matrix model side, we can apply the Polchinski's scattering equation [23]

$$\epsilon_+(\tilde{t} - \tilde{\phi}) = \epsilon_-\left(\tilde{t} - \tilde{\phi} - \log\left(\epsilon_+ \frac{(\tilde{t}-\tilde{\phi})}{2}\right)\right), \tag{3.12}$$

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where ϵ_+ and ϵ_- are the incoming and outgoing deformations of the Fermi surface defined previously in (3.4). This equation states that an incoming wave completely turns into the outgoing one by the reflection with a time-delay represented by the $-\log\epsilon_+$ term in (3.12).

We can express excitations from (3.7)

$$\epsilon_+ = 2\mu^{1/b^2} e^{(1-1/b^2)(\tilde{t}-\tilde{\phi})}(1+\delta_+(\tilde{t}-\tilde{\phi}))\,, \qquad \epsilon_- = 2\mu e^{(b^2-1)(\tilde{t}+\tilde{\phi})}(1+\delta_-(\tilde{t}+\tilde{\phi}))\,. \quad (3.13)$$

To make the expression simple, we can introduce

$$\tilde{\delta}_-(x) = \delta_-\left(\frac{x}{b^2} - \frac{1}{b^2}\log\mu\right). \quad (3.14)$$

Then the scattering equation (3.12) becomes

$$(1+\delta_+(x))^{b^2} = 1 + \tilde{\delta}_-\left(x - b^2\log(1+\delta_+(x))\right). \quad (3.15)$$

We can solve (3.15) recursively up to the order $O(\tilde{\delta}_-^3)$,

$$\begin{aligned}\delta_+ = {} & \frac{1}{b^2}\tilde{\delta}_- + \left(-\frac{1}{b^2}\tilde{\delta}_-\tilde{\delta}'_- + \frac{1-b^2}{2b^4}\tilde{\delta}_-^2\right) + \\ & + \left(\left(\frac{1}{6b^6} - \frac{1}{2b^4} + \frac{1}{3b^2}\right)\tilde{\delta}_-^3 + \frac{1}{b^2}\tilde{\delta}_-\tilde{\delta}'^2_- + \left(\frac{3}{2b^2} - \frac{1}{b^4}\right)\tilde{\delta}_-^2\tilde{\delta}'_- + \frac{1}{2b^2}\tilde{\delta}_-^2\tilde{\delta}''_-\right). \quad (3.16)\end{aligned}$$

Notice that the leading relation $\delta_+(x) \sim \frac{\delta_-(x/b^2+const.)}{b^2}$ tells us that the incoming wave with energy $\tilde{E}$ will be shifted into the energy $b^2\tilde{E}$ due to the moving wall. The relation (3.16) shows the $1\to 1, 1\to 2$ and $1\to 3$ scattering[11] of closed strings. As we will show in the last of this section, it is also possible to find the exact solution to (3.15).

In order to compare these results with those of the string theory scattering amplitudes in the background (1.4) , we would like to perform the Lorentz transformation (1.8). The massless scalar field η can be written in terms of the deformation of Fermi surface by using (3.5)

$$\begin{aligned}(\partial_t - \partial_\phi)\eta(t,\phi) &= \pi^{-1/2}\Delta_+(t-\phi)\,, \\ (\partial_t + \partial_\phi)\eta(t,\phi) &= -\pi^{-1/2}\Delta_-\left(t+\phi+\frac{(\log\mu)}{b}\right), \quad (3.17)\end{aligned}$$

where $\Delta_\pm$ is defined by

$$\begin{aligned}\Delta_+(y) &= 2b\mu^{1/b^2}e^{-qy}\cdot\delta_+(by)\,, \\ \Delta_-(y) &= \frac{2}{b}\mu^{1/b^2}e^{-qy}\cdot\tilde{\delta}_-(by)\,. \quad (3.18)\end{aligned}$$

By substituting (3.18) into (3.16), we can find the scattering equation in the original frame

$$\Delta_+ = \Delta_- - \frac{1}{4}\mu^{-1/b^2}(e^{qy}\Delta_-^2)' + \frac{b}{24}\mu^{-2/b^2}(e^{2qy}\Delta_-^3)' + \frac{1}{24}\mu^{-2/b^2}(e^{2qy}\Delta_-^3)''\,. \quad (3.19)$$

[11] The term '$n\to m$ scattering' means that the process with n incoming and m outgoing particles.

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The quantization of η can be done as

$$\eta = \frac{i}{2\pi^{1/2}} \int_{-\infty}^{\infty} \frac{dE}{E} \left(a_E e^{iE(t-\phi)} + \tilde{a}_E e^{iE(t+\phi)} \right) . \tag{3.20}$$

The creation and annihilation operator satisfy (we follow the convention in [23])

$$[a_E, a_{E'}] = [\tilde{a}_E, \tilde{a}_{E'}] = -E \cdot \delta(E + E') . \tag{3.21}$$

a_E $(E > 0)$ (or $\tilde{a}_E$ $(E > 0)$) represents a creation operator of incoming (or outgoing) particle. Then the $\Delta_\pm$ can be expressed as

$$\begin{aligned} \Delta_+(y) &= -\int \frac{dE}{E}\, a_E\, e^{iEy} , \\ \Delta_-(y) &= \int \frac{dE}{E}\, \tilde{a}_E\, e^{iEy} \mu^{-\frac{i}{b}E} . \end{aligned} \tag{3.22}$$

By plugging (3.22) in (3.19), we obtain

$$\begin{aligned} -\mu^{\frac{i}{b}E} \cdot a_E = \tilde{a}_E &- \frac{i}{4}\mu^{-1} E \int dE'\, \tilde{a}_{E'} \cdot \tilde{a}_{E-E'+iq} + \\ &+ \frac{1}{24}\mu^{-2}(ibE - E^2) \int dE' dE''\, \tilde{a}_{E'} \cdot \tilde{a}_{E''} \cdot \tilde{a}_{E-E'-E''+2iq} . \end{aligned} \tag{3.23}$$

The first term in the right-hand side represents the reflection amplitude (or two point function) and is precisely the same as the one (2.12) obtained in the world-sheet computation after we multiply the previous leg factors in (2.8)

$$\frac{\Gamma(iP/b)}{\Gamma(-iP/b)} \cdot \frac{\Gamma(ibP)}{\Gamma(-ibP)} , \tag{3.24}$$

and perform a scaling $\mu \to \mu\gamma(b^2)$. As we have explained in section 2, each Γ function ratio in (3.24) comes from the incoming or outgoing process, respectively. In this way, we can read off S-matrices from (3.23) including the leg factor (3.24)

$$\begin{aligned} S^{(2)}_{1\to1}(E_1, E_2) &= -\delta(E_1 + E_2) \cdot \mu^{-iE_1/b} \cdot \frac{\Gamma(iE_1/b)}{\Gamma(-iE_1/b)} \cdot \frac{\Gamma(-ibE_2)}{\Gamma(+ibE_2)} E_1 , \\ S^{(3)}_{1\to2}(E_1, E_2, E_3) &= \frac{i}{2}\delta(E_1 + E_2 + E_3 + iq) \cdot \\ &\quad \cdot \mu^{-1-iE_1/b} \cdot \frac{\Gamma(iE_1/b)}{\Gamma(-iE_1/b)} \frac{\Gamma(-ibE_2)}{\Gamma(+ibE_2)} \frac{\Gamma(-ibE_3)}{\Gamma(+ibE_3)} E_1 E_2 E_3 , \\ S^{(4)}_{1\to3}(E_1, E_2, E_3, E_4) &= -\frac{1}{4}\delta(E_1 + E_2 + E_3 + E_4 + 2iq) \cdot \\ &\quad \cdot \mu^{-2-iE_1/b} \cdot \frac{\Gamma(iE_1/b)}{\Gamma(-iE_1/b)} \frac{\Gamma(-ibE_2)}{\Gamma(+ibE_2)} \frac{\Gamma(-ibE_3)}{\Gamma(+ibE_3)} \frac{\Gamma(-ibE_4)}{\Gamma(+ibE_4)} \cdot \\ &\quad \cdot (ib - E_1) E_1 E_2 E_3 E_4 . \end{aligned} \tag{3.25}$$

As we show the details in appendix A, we can see that these amplitudes from the matrix model precisely agree with the string theory results computed in [54].

It is also possible to solve the scattering equation (3.15) exactly by generalizing the method developed in [25]. To find the exact solution, we first consider the infinitesimal variation of $\delta_\pm(x)$ and take the Fourier transformation. Then we obtain the solution to (3.15)

$$\delta_+(x) = \frac{1}{b^2}\sum_{n=1}^{\infty}\frac{\Gamma(-\partial_x+\frac{1}{b^2})}{n!\cdot\Gamma(-\partial_x+\frac{1}{b^2}+1-n)}\cdot(\tilde{\delta}_-(x))^n\,. \tag{3.26}$$

Plugging (3.18) into (3.26) we get in the end

$$\Delta_+(y) = \sum_{n=1}^{\infty}\left(\frac{b\mu^{-1/b^2}}{2}\right)^{n-1}\cdot\frac{\Gamma(-\frac{1}{b}\partial_y+1)}{n!\cdot\Gamma(-\frac{1}{b}\partial_y+2-n)}\cdot\left(e^{(n-1)qy}\Delta_-(y)^n\right). \tag{3.27}$$

It is easy to see that the specific terms in (3.27) of $n=1,2,3$ reproduce (3.19). It is natural to believe that these agreements go over to general $n\to m$ scattering amplitudes as was true [54] in the usual vacuum (1.3) (i.e. the spacial case $b=1$). In the appendix B we also estimated the free energy at tree level and that also agrees with the scaling behavior predicted from string theory. In this way we have confirmed that the matrix model with the Fermi surface (3.1) reproduces the string theory S-matrices in the background (1.4).

3.2 Brief comments on type 0 string cases

It is also possible to extend the above results to the 2d type 0 string [9, 10] in order to consider a non-perturbatively sensible theory (only in this subsection we set $\alpha'=1/2$).

In the type 0B case, there are two copies of the Fermi surface (3.1). We can choose the parameter μ independently for each of the two surfaces and write them as μ_1 and μ_2. Then in this background, there are a non-zero tachyon field T and RR-scalar field C given by

$$T=(\mu_1+\mu_2)\cdot e^{(b^2-1)\tilde{X}^0+(1+b^2)\tilde{\phi}}\,,\qquad C=(\mu_1-\mu_2)\cdot e^{(b^2-1)\tilde{X}^0+(b^2-1)\tilde{\phi}}\,, \tag{3.28}$$

corresponding to the symmetric and asymmetric part with respect to the exchange of the two Fermi sea.

If we consider the type 0A case in the RR-flux background, things become more non-trivial. The equation (3.1) is no more consistent with the time-evolution since the hamiltonian is given [10, 55] by that of the deformed matrix model [56]

$$2H=p^2-x^2+\frac{M}{x^2}\,,\qquad M\equiv q^2-\frac{1}{4}\,, \tag{3.29}$$

where the integer q represents the background RR-flux. It is useful to notice the conserved quantities[12]

$$\begin{aligned} W_+ &= e^{-2t}\left((p+x)^2+\frac{M}{x^2}\right),\\ W_- &= e^{2t}\left((p-x)^2+\frac{M}{x^2}\right). \end{aligned} \tag{3.30}$$

[12] Also notice the relation $W_+W_-=4(M+H^2)$.

Then it is natural to expect the Fermi surface (3.1) in bosonic string is now replaced by

$$W_+^{b^2} W_- = \mu'^2 \,, \tag{3.31}$$

in the 0A model. When there is no RR-flux $M \simeq 0$, it is obvious that the parameter μ' corresponds to that of the tachyon field (setting $\mu' = \mu$ in (1.9)) because the equation (3.31) becomes the same as (3.1). The comparison with the string theory results go over in the same way. In the non-zero RR-flux cases, the precise relation between μ' and μ will be a bit complicated and will be q dependent.

4. Matrix model and time-like Liouville theory

It is possible to add the time-like Liouville potential term

$$S_1 = \nu \int d\sigma^2 e^{-2bX^0} \,, \qquad \text{or} \qquad S_2 = \nu' \int d\sigma^2 e^{2X^0/b} \,, \tag{4.1}$$

to our model defined by (1.4) and (1.5) at least perturbatively. Note that we can put the term (4.1) in addition to the conventional Liouville term because the two CFTs , i.e. time and space-like ones are decoupled.

The time X^0 part of this kind of CFT was considered in [15, 31] by assuming the analytical continuation from the usual space-like Liouville theory. The model is obviously a basic example of rolling closed string tachyon condensation. Though there are several evidences that such a treatment gives sensible results, their properties are far from well-understood. For example, the two potentials in (4.1) are at least formally dual to each other if we extend the result for usual space-like Liouville theory to our time-like case. However, this looks rather strange since the two tachyon fields behave oppositely.

4.1 Matrix model dual of time-like Liouville theory

On the other hand, if we know its matrix model dual, we can define such a theory non-perturbatively. We can rewrite (4.1) as deformations[13] of Fermi surface as we have done previously using (3.4)

$$\epsilon'_- = \nu e^{-(1+b^2)(\tilde{t}+\tilde{\phi})} = \nu e^{-(1+b^2)\tilde{t}} |x|^{1+b^2} \,, \qquad \epsilon'_+ = \nu' e^{-\frac{1+b^2}{b^2}(\tilde{t}+\tilde{\phi})} = \nu' e^{+\frac{1+b^2}{b^2}\tilde{t}} |x|^{\frac{1+b^2}{b^2}} \,. \tag{4.2}$$

These two perturbations of Fermi surface represent the background tachyon fields, i.e. the Lorentz transformation of (4.1)

$$T'_- = \nu e^{-(1+b^2)\tilde{X^0} + (1-b^2)\tilde{\phi}} \,, \qquad T'_+ = \nu' e^{\frac{1+b^2}{b^2}\tilde{X^0} + \frac{b^2-1}{b^2}\tilde{\phi}} \,. \tag{4.3}$$

We can argue that the Fermi surface[14] is now given by

$$(p-x)(-p-x)^{b^2} = \mu e^{-(1-b^2)\tilde{t}} + \nu(-p-x)^{2b^2} e^{-(1+b^2)\tilde{t}} \,, \tag{4.4}$$

[13]In this case the asymptotic geometry is no longer $p = \pm x$. Thus we expect much larger deformations of Fermi surface than those considered in the previous sections. Though in such situation we may even allow the modification of the potential, we will not consider this possibility in this paper.

[14]The special case $b = 1$ has been discussed in [28] from the viewpoint of closed string tachyon condensation and cosmology.

by considering[15] a suitable deformation of (3.9). We assume $p + x < 0$, $p - x > 0$ and $0 < b < 1$, and consider only bosonic string case, though the generalization to type 0 case is possible as in section 3.2. Indeed, the asymptotic tachyon field[16] for (4.4) found from (3.4) is given by the sum of $T_\pm$ and T'_-. It also deserves our attention that when $\mu = 0$ we can exactly regard (4.4) as the analytical continuation $b \to ib$ of (3.1).

The time evolution of this Fermi sea can be summarized as follows. At an early time $t \to -\infty$, the Fermi sea is completely pushed into the infinity and thus there is no spacetime. Then the Fermi sea begins to appear as the closed string tachyon field T'_- becomes smaller. Finally for a large positive t, the Fermi surface approaches the previous one (3.1) and eventually at $t = \infty$ the spacetime looks like a linear dilaton background. Notice that this shows that the other tachyon field T'_+ is not relevant for this matrix model background.

4.2 Spacetime geometry from matrix model

Next we want to check if the 2d spacetime obtained from (4.4) is indeed the same as what we expect from the string theory side. This is much more non-trivial than the previous case (3.1) since the asymptotic behavior at the early time is rather different from the canonical one $p = \pm x$ due to the second term. To see this it is helpful to derive the corresponding collective field theory [51] and try to find how the spacetime looks like. Intuitively, the infinitely long spacial direction of the 2d spacetime is dynamically generated from the infinitely extended Fermi surface. Fluctuations on the Fermi surface correspond to the collective excitations of the fermions and this is conveniently described by the collective field theory [51]. The collective field φ is originally defined by the density of eigenvalues

$$\varphi(x, \tilde{t}) = \mathrm{Tr}\, \delta(x - \Phi(\tilde{t}))\,. \tag{4.6}$$

A fluctuation from its classical value $\varphi_0 = \frac{1}{2\pi}(p_+ - p_-)$ corresponds to a massless scalar field η (or tachyon field T in bosonic string via $T = g_s \cdot \eta$). Therefore one way to know the properties of the spacetime is to investigate propagations of fluctuations on the Fermi surface. As pointed out in [57] (see also [29, 30]), we can extract an effective geometry of the spacetime by computing the kinetic term of η at the quadratic order in the collective field theory, given by

$$S_{(2)} = \int dt \frac{dx}{p_+ - p_-} \left[(\partial_{\tilde{t}}\eta)^2 + (p_+ + p_-)\partial_{\tilde{t}}\eta \partial_x \eta + p_+ p_- (\partial_x \eta)^2 \right]. \tag{4.7}$$

[15] However, this form (4.4) cannot be the unique choice. For instance, we can assume another Fermi surface

$$(p - x)^{1/b}(-p - x)^b = \mu^{1/b} e^{-(1/b - b)\tilde{t}} + \nu^{1/b}(-p - x)^{2b} e^{-(1/b + b)\tilde{t}}\,. \tag{4.5}$$

This will also have the same properties as (4.4) within the discussions in this section since both have the same asymptotic behavior. To find the unique Fermi surface for the string theory (1.9), we need to compare physical quantities explicitly as we have done in the previous section. In this paper we will not go into that detail.

[16] Here we omit the detailed coefficients in front of μ and ν.

Here again p_+ and p_- denote the upper and lower branches of Fermi surface in the (x, p) plane. The 'effective metric' can be found by just comparing (4.7) with the standard expression $\sim \sqrt{g} g^{\mu\nu} \partial_\mu \eta \partial_\nu \eta$ up to the conformal transformation.[17]

Let us apply this method to our example. We can conveniently choose the spacial coordinate σ as follows

$$-p - x = \mu^{\frac{1}{1+b^2}} e^{\sigma}, \qquad p - x = \mu^{\frac{1}{1+b^2}} e^{-b^2\sigma + (b^2-1)\tilde{t}} + \nu \mu^{\frac{1}{1+b^2}} e^{b^2\sigma - (1+b^2)\tilde{t}}. \tag{4.8}$$

We can find two solutions of p to (4.4) for fixed x. We parameterize the two branches by $p_+ = p(\sigma, \tilde{t})$ and $p_- = p(\tilde{\sigma}, \tilde{t})$ by introducing another function $\tilde{\sigma}(\sigma, \tilde{t})$ such that $x(\sigma) = x(\tilde{\sigma})$. The parameters take the values

$$-\infty < \sigma \leq \sigma_0(\tilde{t}), \qquad \sigma_0(\tilde{t}) \leq \tilde{\sigma} < \infty, \tag{4.9}$$

where σ_0 is a time-dependent function and behaves like $\sigma_0(\tilde{t}) \sim -\frac{1+b^2}{1-b^2}\tilde{t}$ for large $\tilde{t}$.

Now, we can rewrite the effective field theory (4.7) in terms of the coordinate $(\tilde{t}, \tilde{\sigma})$. Let us consider the asymptotic geometry, i.e. we assume that $|\tilde{t}|$ and $\tilde{\sigma}$ are large, to make the computations simple. When the two conditions (the first one just corresponds to (4.9))

$$(1 - b^2)\tilde{\sigma} + (1 + b^2)\tilde{t} > 0, \qquad (1 + b^2)\tilde{\sigma} + (1 - b^2)\tilde{t} > 0, \tag{4.10}$$

are satisfied, the first and second exponential terms (i.e. the ν independent ones) in the right-hand side of (4.8) are dominant for the large values of $|\tilde{t}|$ and $\tilde{\sigma}$. In this case, $(\tilde{t}, \tilde{\sigma})$ coincides with the coordinate $(\tilde{X^0}, -\tilde{\phi})$ in the string theory side. Indeed, the kinetic term of (4.7) takes the standard form $\sim (\partial_{\tilde{t}}\eta)^2 - (\partial_{\tilde{\sigma}}\eta)^2$. It is also possible to see that for the other values than (4.10), the effective 'metric' obtained from (4.7) degenerates into that of a line and thus this does not contribute to the spacetime geometry. Thus we can conclude that the spacetime is given by the region (4.10) or equally

$$\{(\tilde{X^0}, \tilde{\phi})|\ (1 - b^2)\tilde{\phi} - (1 + b^2)\tilde{X^0} < 0, \ \ (1 + b^2)\tilde{\phi} - (1 - b^2)\tilde{X^0} < 0\}. \tag{4.11}$$

Indeed, this is consistent with the expectation in the world-sheet theory side. The two conditions in (4.11) correspond to the tachyon walls T'_- in (4.3) and $T_\pm$ in (3.8), respectively. In other words, if we return to the the original frame, the condition (4.11) just means the upper bound for X^0 and the lower bound for ϕ. It is again confirmed that the other tachyon field T'_+ does not contribute in this background. This will be a good lesson when we analyze the time-like Liouville CFT.

Then one may ask what is the matrix model configuration dual to the tachyon field T'_+. If we remember the dual equivalent expression of the Fermi surface (3.9), we can easily identify it with

$$(-p - x)(p - x)^{1/b^2} = \mu^{1/b^2} e^{-(1/b^2-1)\tilde{t}} + \nu'(p - x)^{2/b^2} e^{(1+1/b^2)\tilde{t}}. \tag{4.12}$$

The previous arguments can also be applied to this case similarly. If we simply assume $\mu = 0$, then the two different backgrounds defined by T'_- and T'_+ correspond to the upper and lower region divided by the surface $p - x = \nu e^{-(1+b^2)\tilde{t}} |p + x|^{b^2}$, respectively.

[17]This means that we can always find a coordinate where the metric is flat as noted in [57]. Here we use the effective metric to see if the coordinate we assumed is singular in that region.

5. Ground ring and possible relations to non-compact Calabi-Yau

So far we have investigated the equivalence between the 2d string theory in our specific backgrounds and its dual matrix model description by looking at the properties of the tachyon field in the 2d spacetime. There is another helpful proposal [58] that we can directly relate the Fermi surface to the ring structure, so called ground ring, of BRST invariant operators at ghost number zero. In the ordinary static $c = 1$ vacuum this is simply given[18] by

$$\mathbf{x}\mathbf{y} = \mu\,, \tag{5.1}$$

as proposed in [58] and proved in [10] explicitly ($\mathbf{x}$ and $\mathbf{y}$ are the ground ring generators and will be defined below more generally). Indeed this agrees with the Fermi surface equation (1.3) after a rather trivial change of basis. Let us apply this idea to our examples.[19]

When the value b^2 takes rational values $0 < \frac{p}{q} \leq 1$ (p and q are coprime integers), we can write the Fermi surface equation (3.1) in the form

$$W_{0,1}^q W_{1,0}^p = \mu^q\,, \tag{5.2}$$

using the conserved quantities (3.11). This strongly implies that the ground ring structure [58] in our background (1.4) will be

$$\mathbf{x}^q \mathbf{y}^p = \mu^q\,, \tag{5.3}$$

where $\mathbf{x} = \mathbf{a}\bar{\mathbf{a}}$ and $\mathbf{y} = \mathbf{b}\bar{\mathbf{b}}$ are the ground ring generators. We can write them explicitly via the Lorentz transformation (we show only the ones in the left-moving sector)

$$\mathbf{a} = \left(cb + \sqrt{\frac{q}{p}}\,\partial(\phi + iX)\right) e^{\sqrt{\frac{p}{q}}(iX-\phi)}\,, \qquad \mathbf{b} = \left(cb + \sqrt{\frac{p}{q}}\,\partial(\phi - iX)\right) e^{-\sqrt{\frac{q}{p}}(iX+\phi)}\,, \tag{5.4}$$

where $X = iX^0$ is the euclidean time. Obviously for $p = q = 1$ this statement is reduced to the basic result (5.1). For general p and q, we will be able to show this relation almost in the same way.

These expressions (5.4) are formally the same as the ground ring generators for the (p, q) minimal string in the coulomb gas description [60, 61]. Nevertheless the ground ring structure (5.3) for our non-minimal case is different from that of the minimal string found in [59] because there are screening operators in the minimal model case. On the other hand, if we consider another one (4.4) (or (4.5)) corresponding to the time-like Liouville potential, we obtain the relation at $\mu = 0$

$$\mathbf{y}^p \cdot (\mathbf{x}^q - \nu^q\, \mathbf{y}^p) = 0\,. \tag{5.5}$$

This looks very close to the one in the minimal string [59, 62], except the factor $\mathbf{y}^p$. This may be natural since we now have the Liouville potential (or screening operator) in the matter CFT as in the minimal case. We would like to leave the details on this issue for future work.

[18] Here we mean that this relation does hold for the on-shell tachyon states as clarified in [59]. The author thank David Shih for explaining this point.

[19] The author especially thank Davide Gaiotto and Cumrun Vafa for very useful suggestions and comments on this section.

As pointed out in [10], the values of the ground ring elements are also directly related to the charges carried by decaying D0-branes.[20] Generalizing this analysis to our case, we can show that the expectation value of $\mathbf{x}$ and $\mathbf{y}$ on a decaying D0-brane discussed in section 2.1 is given by (up to a constant)

$$\langle \mathbf{x} \rangle = \mu_B \mu^{\frac{1}{2}} \,, \qquad \langle \mathbf{y} \rangle = \mu_B^{\frac{1}{b^2}} \mu^{\frac{1}{2b^2}} \,, \tag{5.6}$$

employing the boundary Liouville theoretic results [42, 43]. The μ_B dependence of (5.6) is indeed consistent[21] with the expectation values of $W_{0,1}$ and $W_{1,0}$ for the trajectory $-\lambda(t) = \mu_B e^{-bt} + \tilde{\mu}_B e^{t/b}$, where we have included the dual cosmological constant $\tilde{\mu}_B = \mu_B^{1/b^2}$. It would also be intriguing to study the other kind of D-branes (or non-compact branes) in these spaces and compare them with the dual 2d dimensional string in order to understand the open-closed duality [62, 64].

As is well known, the $c = 1$ string at the self-dual radius is equivalent to the topological string (B-model) on the conifold [65] (refer to [66]–[69] for more general backgrounds obtained from the quotients or perturbations of $c = 1$ string, and also refer to [62] for modern perspectives.). Thus we may expect that our backgrounds, when suitably compactified, will also be dual to the topological string on specific non-compact Calabi-Yau manifolds. After we wick-rotate the time into the Euclidean one, we can impose the periodicity[22] $X \sim X + 2\pi\sqrt{pq}$ (i.e. the radius $R = \sqrt{pq}$ in the $\alpha' = 1$ unit) since the coupling constant $g_s \propto e^{-iqX}$ respects this.[23] The most plausible speculation will be that this compactified background is dual to the non-compact Calabi-Yau manifold defined by

$$\mathbf{x}^q \mathbf{y}^p + \mathbf{w}\mathbf{z} = \mu^q \,. \tag{5.7}$$

We can choose the corresponding ground ring generators so that the momentum and winding number obey the standard quantization rule

$$\mathbf{x} = \mathbf{a}\bar{\mathbf{a}} \,, \qquad \mathbf{y} = \mathbf{b}\bar{\mathbf{b}} \,, \qquad \mathbf{w} = \mathbf{a}^q \bar{\mathbf{b}}^p \,, \qquad \mathbf{z} = \mathbf{b}^p \bar{\mathbf{a}}^q \,. \tag{5.8}$$

It will be possible to find a similar algebraic equation for type 0 string case (see [10, 71, 12, 72] for relevant discussions on the $\hat{c} = 1$ string).

It may be helpful to compare this with the known ground ring for the $c = 1$ string at the radius $R = \frac{r}{s}$ (r and s are coprime integers), given by

$$(\mathbf{x}\mathbf{y})^r + (\mathbf{w}\mathbf{z})^s = \mu' \,, \tag{5.9}$$

[20] In the papers [63], another definition of the conserved charges carried by the D0-branes was considered. This may also lead to similar interesting results in our case.

[21] Note that here we only discuss the 'half S-brane' [36]–[38]. For more general boundary interactions like an analogue of the 'cosh' brane [35], we will expect a non-trivial renormalization of μ_B as is so in the $c = 1$ CFT.

[22] This is a different compactification radius than the one $R = \sqrt{\frac{p}{q}}$ in the Coulomb-gas representation of the minimal model. This is because the latter has the screening operators. Our model does not have such operators and thus this smaller radius is not consistent with g_s.

[23] A similar compactification in a time-like direction also recently discussed in a matrix model dual of harmonic oscillator [70].

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as found in [68] . This is obviously a different background from ours. However, it is curious that in the special case of the common radius $R = q \in \mathbb{Z}$, this equation (5.9) agrees with ours (5.7).

6. Conclusions and discussions

In this paper we have discussed a matrix model dual of the 2d string theory with a time-like linear dilaton matter. This may be called as a non-minimal $c < 1$ non-critical string. Compared with the standard minimal model case, we can allow irrational values of the central charge. After the Lorentz transformation this background is equivalent to the usual $c = 1$ string with a non-standard and time-dependent Liouville potential. We identified the corresponding time-dependent Fermi surface in the dual $c = 1$ matrix model. We compared the tree level scattering S-matrices in the matrix model with those computed in 2d string theory and found a perfect agreement. It would be interesting to find a precise matrix model dual description beyond the tree level. Notice that in other words, we have discussed how to realize the Lorentz transformation, which is only manifest in 2d closed string theory, from the viewpoint of its holographic dual open string theory defined in the lower dimension. We also proposed an equivalent topological string description on a series of specific non-compact Calabi-Yau manifolds given by (5.7).

Another interesting quantity in the matrix model which we may be able to compare with string theory will be the macroscopic loop operator $\log(\mu_B + \Phi)$ [73] (see also the review [8]). As shown in [74], it is equivalent to the FZZT-brane with Dirichlet boundary condition in the time-direction when we assume the usual vacuum $b = 1$. For generic b, however, this does not look straightforward because there is a linear dilaton in the time direction and its Dirichlet boundary condition is not well-defined. Since the loop operator itself is well defined even in time-dependent background, this will be an intriguing future problem.[24] A related question will be the D-brane spectrum[25] in the dual non-compact Calabi-Yau (5.7) and its relation to the boundary states in our background (1.4).

We also noticed that the matrix model description predicts a series of new boundary states in our backgrounds of two dimensional string theory. This is a generalization of the known boundary states for the rolling tachyon $T(t) \sim \cosh(t)$ [35] in our time-like linear dilaton case.

Furthermore, we discussed the 2d string theory whose matter part (or time part) is given by a time-like Liouville theory. We considered the dual matrix model configuration. Interestingly, we noticed that the dual cosmological constant does not automatically appear when the original cosmological constant is non-zero.[26] Also we find that the ground ring structure looks very similar to that of the minimial string. To understand better the duality

[24]For example, we can write down the deformation of Fermi surface due to the loop operator as in [75]. Though we can read off from this the one-point function of the corresponding D-brane boundary state, its explicit form does not look so simple except the ordinary case $b = 1$.

[25]In the minimal model case, we can associate the moduli of FZZT-branes with a Riemann-surface in non-compact Calabi-Yau spaces [62, 59] at tree level.

[26]This may also solve a similar puzzle in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ WZW model at the level $0 < k < 2$ and its sine-Liouville dual noticed in [17].

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between the time-like Liouville theory and our matrix model background, we will need to compare dynamical quantities like scattering amplitudes. Even though it is not clear if we can define scattering processes in the string theory side of the time-like Liouville theory, it seems that we can consider an incoming wave in the matrix model background and try to follow the time-evolution. This issue will also deserve a future study.

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A. Comparison of S-matrices in 2D string theory

Here we summarize the results of S-matrix in 2d String Theory. First we follow the notation of [54] i.e. $\alpha' = 2$ and the Liouville potential is $\mu \int d\sigma^2 e^{-\sqrt{2}b\phi}$. The vertex operators are given[27] by $e^{ikX+(-Q/\sqrt{2}+|k|)\phi}$ $(Q = b + 1/b)$ and X is now euclidean. We define the 'leg factor' (see (2.8))

$$\Delta(k) = \frac{\Gamma(1-\frac{\sqrt{2}k}{b})}{\Gamma(\frac{\sqrt{2}k}{b})} \qquad (k>0)\,, \qquad \Delta(k) = \frac{\Gamma(1+\sqrt{2}bk)}{\Gamma(-\sqrt{2}bk)} \quad (k<0)\,. \tag{A.1}$$

The S-matrix of three particles are given by

$$S^{(3)}(k_1,k_2,k_3) = \delta\left(k_1+k_2+k_3+\frac{q}{\sqrt{2}}\right)\cdot(\mu\gamma(b^2))^s \cdot \prod_{i=1}^{3}(-\pi\cdot\Delta(k_i))\,, \tag{A.2}$$

where s is the number of insertions of the Liouville potential term so that it satisfies the momentum conservation

$$\sum_{i=1}^{3}|k_i| - \sqrt{2}bs = \frac{1}{\sqrt{2}}Q\,. \tag{A.3}$$

The four point function is

$$\begin{aligned} S^{(4)}(k_1,k_2,k_3,k_3) \;=\; & \delta(k_1+k_2+k_3+k_4+\sqrt{2}q)\cdot(\mu\gamma(b^2))^s\cdot\prod_{i=1}^{4}(-\pi\cdot\Delta(k_i))\,\cdot \\ & \cdot\left[\frac{1}{\sqrt{2}b}\left(\left|k_1+k_2+\frac{q}{\sqrt{2}}\right|+\left|k_1+k_3+\frac{q}{\sqrt{2}}\right|+\left|k_1+k_4+\frac{q}{\sqrt{2}}\right|\right)-\frac{1+b^2}{2b^2}\right], \end{aligned} \tag{A.4}$$

where s is given by

$$\sum_{i=1}^{4}|k_i| - \sqrt{2}bs = \sqrt{2}Q\,. \tag{A.5}$$

[27] Here we have shifted k in [54] by $\alpha = 0$.

Let us compare these results of $1 \to 2$ and $1 \to 3$ scattering with those in the matrix model computed in section 3. To match the convention we have to return to the Minkowski signature with $\alpha' = 1$ unit performing the scaling

$$\sqrt{2}k \to -iE\,. \tag{A.6}$$

Then they are written as follows[28]

$$\begin{aligned}
S^{(3)}_{1\to 2} &= \frac{i}{b} \cdot \delta(E_1 + E_2 + E_3 + iq) \cdot (\mu\gamma(b^2))^{-1-iE_1/b} \cdot (-\pi)^3 \times \\
&\quad \times \frac{\Gamma(1+iE_1/b)}{\Gamma(-iE_1/b)} \frac{\Gamma(1-ibE_2)}{\Gamma(ibE_2)} \frac{\Gamma(1-ibE_3)}{\Gamma(ibE_3)} \\
&= \delta(E_1 + E_2 + E_3 + iq) \cdot (\mu\gamma(b^2))^{-1-iE_1/b} \cdot (-\pi)^3 \cdot E_1 E_2 E_3 \times \\
&\quad \times \frac{\Gamma(iE_1/b)}{\Gamma(-iE_1/b)} \frac{\Gamma(-ibE_2)}{\Gamma(ibE_2)} \frac{\Gamma(-ibE_3)}{\Gamma(ibE_3)}\,, \\
S^{(4)}_{1\to 3} &= \frac{i}{b} \cdot \delta(E_1 + E_2 + E_3 + E_4 + 2iq) \cdot (\mu\gamma(b^2))^{-2-iE_1/b} \cdot (-\pi)^4 \cdot (-1 - iE_1/b) \times \\
&\quad \times \frac{\Gamma(1+iE_1/b)}{\Gamma(-iE_1/b)} \frac{\Gamma(1-ibE_2)}{\Gamma(ibE_2)} \frac{\Gamma(1-ibE_3)}{\Gamma(ibE_3)} \frac{\Gamma(1-ibE_4)}{\Gamma(ibE_4)} \\
&= \delta(E_1 + E_2 + E_3 + E_4 + 2iq) \cdot (\mu\gamma(b^2))^{-2-iE_1/b} \cdot (-\pi)^4 \cdot (ib - E_1) E_1 E_2 E_3 E_4 \times \\
&\quad \times \frac{\Gamma(iE_1/b)}{\Gamma(-iE_1/b)} \frac{\Gamma(-ibE_2)}{\Gamma(ibE_2)} \frac{\Gamma(-ibE_3)}{\Gamma(ibE_3)} \frac{\Gamma(-ibE_4)}{\Gamma(ibE_4)}\,.
\end{aligned} \tag{A.7}$$

In the end we can show that the string theory S-matrices exactly agree with those of $c = 1$ matrix model (3.25) taking into account the scaling $\mu \to \mu\gamma(b^2)$ and the field normalization

$$S_{\mathrm{mat}}(t, \phi) = \left(-\frac{i}{2\pi}\right) \cdot S_{\mathrm{string},\alpha'=1}(t, \phi)\,. \tag{A.8}$$

B. Computation of free energy

It will also be useful to find the tree level free energy in the matrix model background. We can estimate the expectation values $v_{n,m}$ of the (classical) w_∞ algebra [50, 6]. Since the Fermi sea extends infinitely, we need a cut off $|x| < \Lambda$. We can explicitly evaluate the classical contributions in the late time $t \gg 1$ as follows

$$\begin{aligned}
v_{n,m} &\equiv e^{(n-m)t} \int_{F-F_0} \frac{dx\,dp}{2\pi} (-p-x)^m (p-x)^n \\
&= -\frac{e^{(n-m)t}}{m+1} \left[\int_a^\Lambda \frac{dx}{2\pi} \left(2\mu^{1/b^2} e^{(1-1/b^2)\tilde{t}} |x|^{-1/b^2} \right)^{m+1} \cdot (2|x|)^n \right] - \\
&\quad - \frac{e^{(n-m)t}}{n+1} \left[\int_a^\Lambda \frac{dx}{2\pi} \left(2\mu e^{(b^2-1)\tilde{t}} |x|^{-b^2} \right)^{n+1} \cdot (2|x|)^m \right] \\
&= C_{n,m} \cdot \mu^{\frac{n+m+2}{1+b^2}} \cdot e^{\left[\frac{2b^2}{1+b^2}(n+1) - \frac{2(m+1)}{1+b^2}\right]\tilde{t}} + (\Lambda \text{ dependent term})\,,
\end{aligned} \tag{B.1}$$

[28] Here we neglect the common factor $\sqrt{2}$ which comes from the delta function normalization. Also we put a factor $\frac{1}{b}$ due to the integration over the zero mode of ϕ, which is not explicitly written in [54].

where F is our Fermi surface and F_0 is the one defined by $p^2 - x^2 = 0$. a is defined to be $a = \frac{1}{2} \cdot b^{\frac{2}{1+b^2}}(1 + 1/b^2)$. The constant C is given by

$$C_{n,m} = \frac{2^{n+m+1}}{\pi((n+1)b^2 - m + 1)} \cdot \left(\frac{b^2}{m+1} \cdot (2a)^{n+1-(m+1)/b^2} - \frac{1}{n+1} \cdot (2a)^{-(n+1)b^2+m+1} \right). \tag{B.2}$$

In particular, the energy of the system is

$$v_{1,1} = C_{1,1} \cdot \mu^{\frac{4}{1+b^2}} \cdot e^{4\frac{b^2-1}{b^2+1}\tilde{t}}. \tag{B.3}$$

On the other hand, in the string theory on the background (1.4) we can estimate the time-dependent energy

$$F(t) \sim (g_s)^{-2} = e^{-2qt-2Q\phi_0}, \tag{B.4}$$

where ϕ_0 is the characteristic value given by $\mu e^{2b\phi_0} = 1$. Then we can see that the both results (B.3) and (B.4) agree with each other because

$$\mu^{\frac{4}{1+b^2}} \cdot e^{4\frac{b^2-1}{b^2+1}\tilde{t}} = \mu^{\frac{4}{1+b^2}} \cdot e^{-2qt-2\frac{q^2}{Q}\phi_0} = e^{-2qt-2Q\phi_0}. \tag{B.5}$$

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